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Algorithms & Data Structures — Exercise sheet 1 — HS 23

The solutions for this sheet are submitted at the beginning of the exercise class on 2 October 2023.

Exercises that are marked by * are challenge exercises. They do not count towards bonus points.

You can use results from previous parts without solving those parts.

Exercise 1.1 *Guess the formula* **(1 point)**.

Consider the recursive formula defined by $a_1 = 2$ and $a_{n+1} = 3a_n - 2$ for $n > 1$. Find a simple closed formula for a_n and prove that a_n follows it using mathematical induction.

Hint: *Write out the first few terms. How fast does the sequence grow?*

Exercise 1.2 *Sum of Cubes* **(1 point)**.

Prove by mathematical induction that for every positive integer n,

$$1^3 + 2^3 + \cdots + n^3 = \frac{n^2(n+1)^2}{4}.$$

Exercise 1.3 *Sums of powers of integers.*

In this exercise, we fix an integer $k \in \mathbb{N}_0$.

(a) Show that, for all $n \in \mathbb{N}_0$, we have $\sum_{i=1}^{n} i^k \leq n^{k+1}$.

(b) Show that for all $n \in \mathbb{N}_0$, we have $\sum_{i=1}^{n} i^k \geq \frac{1}{2^{k+1}} \cdot n^{k+1}$.

Hint: *Consider the second half of the sum, i.e., $\sum_{i=\lceil \frac{n}{2} \rceil}^{n} i^k$. How many terms are there in this sum? How small can they be?*

Together, these two inequalities show that $C_1 \cdot n^{k+1} \leq \sum_{i=1}^{n} i^k \leq C_2 \cdot n^{k+1}$, where $C_1 = \frac{1}{2^{k+1}}$ and $C_2 = 1$ are two constants independent of n. Hence, when n is large, $\sum_{i=1}^{n} i^k$ behaves “almost like n^{k+1}” up to a constant factor.

Exercise 1.4 *Asymptotic growth* **(1 point)**.

Recall the concept of asymptotic growth that we introduced in Exercise sheet 0: If $f, g : \mathbb{N} \to \mathbb{R}^+$ are two functions, then:

- We say that f *grows asymptotically slower than* g if $\lim_{m\to\infty} \frac{f(m)}{g(m)} = 0$. If this is the case, we also say that g *grows asymptotically faster than* f.

Prove or disprove each of the following statements.

(a) $f(m) = 10m^3 - m^2$ grows asymptotically slower than $g(m) = 100m^3$.

(b) $f(m) = 100 \cdot m^2 \log(m) + 10 \cdot m^3$ grows asymptotically slower than $g(m) = 5 \cdot m^3 \log(m)$.

Hint: $\log(m)$ *grows asymptotically slower than* m.

(c) $f(m) = \log(m)$ grows asymptotically slower than $g(m) = \log(m^4)$.

(d) $f(m) = 2^{(0.9m^2+m)}$ grows asymptotically slower than $g(m) = 2^{(m^2)}$.

(e) If f grows asymptotically slower than g, and g grows asymptotically slower than h, then f grows asymptotically slower than h.

Hint: *For any* $a, b : \mathbb{N} \to \mathbb{R}^+$, *if* $\lim_{m\to\infty} a(m) = A$ *and* $\lim_{m\to\infty} b(m) = B$, *then* $\lim_{m\to\infty} a(m)b(m) = AB$.

(f) If f grows asymptotically slower than g, and $h : \mathbb{N} \to \mathbb{N}$ grows asymptotically faster than 1, then f grows asymptotically slower than $g(h(m))$.

Exercise 1.5 *Proving Inequalities.*

(a) By induction, prove the inequality

$$\frac{1}{2} \cdot \frac{3}{4} \cdot \frac{5}{6} \cdot \dots \cdot \frac{2n-1}{2n} \leq \frac{1}{\sqrt{3n+1}}, \quad n \geq 1.$$

(b)* Replace $3n + 1$ by $3n$ on the right side, and try to prove the new inequality by induction. This inequality is even weaker, hence it must be true. However, the induction proof fails. Try to explain to yourself how is this possible?